

A reformulation of the Barrabès-Israel null-shell formalism

Eric Poisson

Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1

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We consider a situation in which two metrics are joined at a null hypersurface. It often occurs that the union of the two metrics gives rise to a Ricci tensor that contains a term proportional to a Dirac δ -function supported on the hypersurface. This singularity is associated with a thin distribution of matter on the hypersurface, and following Barrabès and Israel, we seek to determine its stress-energy tensor in terms of the geometric properties of the null hypersurface. While our treatment here does not deviate strongly from their previous work, it offers a simplification of the computational operations involved in a typical application of the formalism, and it gives rise to a stress-energy tensor that possesses a more recognizable phenomenology. Our reformulation of the null-shell formalism makes systematic use of the null generators of the singular hypersurface, which define a preferred flow to which the flow of matter can be compared. This construction provides the stress-energy tensor with a simple characterization in terms of a mass density, a mass current, and an isotropic pressure. Our reformulation also involves a family of freely-moving observers that intersect the surface layer and perform measurements on it. This construction gives operational meaning to the stress-energy tensor by fixing the argument of the δ -function to be proper time as measured by these observers.

I. INTRODUCTION

Many applications of the general theory of relativity involve two metrics being joined at a common boundary, a hypersurface that partitions spacetime into two distinct regions. It often occurs that the union of the metrics does not form a smooth solution to the Einstein field equation, even when the metric is continuous at the hypersurface. Indeed, a discontinuity in the metric's transverse derivative always produces a singularity in the Riemann tensor. The singularity, however, is sufficiently mild (it is a Dirac δ -function with support on the hypersurface) that it yields itself to a sound physical interpretation.

When the hypersurface is timelike (or spacelike), only the Ricci part of the Riemann tensor is singular, and the singularity can be associated with a material stress-energy tensor supported on the hypersurface. When the hypersurface is null, on the other hand, both the Weyl and Ricci parts of the Riemann tensor are singular; here the singularity in the Ricci tensor can just as well be associated with a material stress-energy tensor, while the singularity in the Weyl tensor is interpreted as an independent impulsive tidal wave. The material present on the hypersurface, which gives rise to a singular Ricci tensor, is often referred to as a *surface layer*, or as a *thin shell*. The dynamics of a timelike or null surface layer, whose world surface coincides with the singular hypersurface, can be of considerable interest and has been the subject of numerous studies.

The standard formalism to describe the physical properties of a timelike (or spacelike) surface layer was produced by Israel more than 35 years ago [1]. This practical formalism allows for efficient computations of the shell's surface stress-energy tensor in terms of the discontinuity of the extrinsic curvature across the hypersurface. The Israel formalism has even been implemented as the package **GRjunction** [2] within the computer algebra program

GrTensorII [3].

Several extensions of the Israel formalism to the case of a null hypersurface were proposed in the following years [4, 5, 6, 7, 8, 9], but a definitive treatment had to wait the 1991 work of Barrabès and Israel [10], who produced a formulation that was much better suited to practical applications. Their formalism was used in several interesting contexts: to model an exploding white hole [11]; to speculate on the possible creation of expanding universes inside black holes [12]; to calculate the gravitational signal that accompanies a body's sudden change of multipole moments [13, 14], or its sudden deceleration to a complete stop [15, 16]; and to calculate the detection of an impulsive null signal by a system of neighbouring test particles [17, 18]. A number of additional applications were considered by Barrabès and Hogan in their extension of the original Barrabès-Israel formalism [19], which was also extended to the case of scalar-tensor theories [20, 21] and the Einstein-Cartan theory [22]. The formalism has been implemented within the computer package **GRjunction** [23].

Our purpose with this paper is to present a reformulation of the Barrabès-Israel formalism for null surface layers. With this new formulation we hope to achieve both a simplification of the computational steps involved in a typical application of the formalism, and a clarification of some of the interpretational aspects of the formalism. While the work presented here does not deviate strongly from the original Barrabès-Israel treatment, we believe that it will ease and encourage further explorations of the physics of null surface layers; as such, our work should constitute a useful addition to the existing literature. In effect, what we set out to do is to make the Barrabès-Israel formalism more user-friendly, and more amenable to straightforward physical interpretation.

The first key aspect of this reformulation of the null-shell formalism is our systematic involvement of the null

generators of the singular hypersurface. The generators are used to define a preferred flow on the hypersurface, to which the flow of matter on the surface layer can be compared. As we shall see, this construction provides the surface stress-energy tensor with a simple characterization in terms of a mass density (which represents a flow of matter along the null generators), a current density (which represents a flow of matter across generators), and an isotropic pressure. The surface stress-energy tensor therefore comes with a clear and meaningful phenomenology.

The null generators also provide us with a preferred family of intrinsic coordinates on the hypersurface. These are adapted to generators, in the sense that two of the coordinates serve to label the generators, while the third coordinate is the generator's parameter. The reformulation makes exclusive use of this family of intrinsic coordinates. While this represents a restriction of the freedom granted by the original treatment of Barrabès and Israel (who allowed for an arbitrary system of intrinsic coordinates), we shall see that this choice greatly simplifies the computational operations that arise in practical applications of the formalism. A particularly important aspect of this simplification is that once the intrinsic coordinates have been constructed, all remaining operations follow a simple algorithm. We admit that the explicit construction of the preferred coordinates can be a difficult task, and that in some cases this might motivate a retreat to the original formulation. In many applications, however, including the ones considered in this paper, the intrinsic coordinates are easily obtained and the remaining operations are entirely straightforward.

The second key aspect of our reformulation is an attempt to give operational meaning to the δ -function structure of the surface layer's stress-energy tensor. In the original treatment, the argument of the δ -function is left arbitrary, and as a result the stress-energy tensor acquires an ambiguity: it is defined up to a multiplicative factor that can vary arbitrarily over the hypersurface. Our reformulation does not eliminate this ambiguity (which is an unavoidable feature of null hypersurfaces), but it characterizes it in a physical, well-motivated way. We introduce, in a neighbourhood of the hypersurface, a family of freely-moving observers that intersect the surface layer and perform measurements on it. This family is arbitrary, and it is this arbitrariness that now generates the ambiguity associated with the surface stress-energy tensor. Once a specific choice of observers has been made, however, all traces of ambiguity disappear. The argument of the δ -function is then fixed to be the observer's proper time, which is set to zero at the hypersurface. This choice confers a robust operational meaning to the surface stress-energy tensor.

Our reformulation of the Barrabès-Israel null-shell formalism is presented in Sec. II of the paper; there we omit all derivations and present the formalism as a ready-to-use recipe. Derivations of the main results are contained in Sec. III. In Sec. IV we examine some issues regarding

the parameterization of the null generators and their impact on the physics of null surface layers. Both Secs. III and IV can be omitted during a first reading of the paper; they contain technical material that are not required in practical applications of the formalism. Finally, in Sec. V we present four simple applications of the formalism, designed to illustrate the methods and highlight some aspects of the physics of null surface layers.

II. NULL-SHELL FORMALISM

We consider a hypersurface Σ that partitions spacetime into two regions \mathcal{M}^\pm . In \mathcal{M}^\pm the metric is $g_{\alpha\beta}^\pm$ when expressed in coordinates x_\pm^α ; as indicated, the coordinate systems on each side of the hypersurface may be different. We assume that the hypersurface is null, and our convention is such that \mathcal{M}^- is in the past of Σ , and \mathcal{M}^+ in its future. We assume also that the hypersurface is singular, in the sense that in a suitable coordinate system defined in a local coordinate neighbourhood that includes Σ , the Riemann tensor is distributional and contains a term proportional to a Dirac δ -function supported on the hypersurface. In general, both the Weyl and Ricci parts of the Riemann tensor are singular. The singularity in the Ricci tensor can be thought to be produced by a material surface layer whose spacetime history coincides with Σ . We wish to give a characterization of this surface layer in terms of a surface stress-energy tensor. We shall here have nothing to say about the singularity in the Weyl tensor and the interesting physics associated with it; for a discussion we refer the reader to the relevant literature (for example, see [4, 5, 6, 10, 19]).

Intrinsic coordinates

Our formulation of the null-shell formalism is based on a preferred family of coordinate systems that are used to represent events on the hypersurface; these coordinates are adapted to the null generators of Σ . Since the generators define a preferred flow on the hypersurface, and since the preferred coordinates give the simplest description of the generators, this type of intrinsic coordinate system will be seen to produce an especially simple description of the surface stress-energy tensor.

The preferred intrinsic coordinates are

$$y^a = (\lambda, \theta^A), \quad (2.1)$$

where the lower-case latin index a runs over the values $(1, 2, 3)$, while the upper-case index A runs over $(2, 3)$; thus $y^1 = \lambda$, $y^2 = \theta^2$, and $y^3 = \theta^3$. We assume that the coordinates y^a are the same on both sides of Σ . We take λ to be an arbitrary parameter on the null generators of the hypersurface, and we use θ^A to label the generators. Thus, to each generator we assign two coordinates θ^A , and to each event on a given generator we assign a

third coordinate λ . Changing λ while keeping θ^A constant produces a displacement along a single generator; changing θ^A while keeping λ constant produces a displacement across generators. It is possible to choose λ to be an *affine* parameter on *one side* of the hypersurface; as we shall see in Sec. IV, however, it is in general not possible to make λ an affine parameter on both sides of Σ . The coordinate system of Eq. (2.1) is not unique, as it leaves the freedom of performing an independent reparameterization of each generator; we shall explore this freedom in Sec. IV.

Tangent vectors

As seen from \mathcal{M}^\pm , Σ is described by a set of parametric relations $x_\pm^\alpha(y^a)$, and using these we can introduce tangent vectors $e_{\pm a}^\alpha = \partial x_\pm^\alpha / \partial y^a$ on each side of the hypersurface. These are naturally segregated into a null vector k_\pm^α that is tangent to the generators, and two spacelike vectors $e_{\pm A}^\alpha$ that point in the directions transverse to the generators. Explicitly,

$$k^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda} \right)_{\theta^A} \equiv e_\lambda^\alpha, \quad e_A^\alpha = \left(\frac{\partial x^\alpha}{\partial \theta^A} \right)_\lambda. \quad (2.2)$$

(Here and below, in order to keep the notation simple, we refrain from using the “ \pm ” label in displayed equations; this should not create any confusion.) By construction, these vectors satisfy

$$k_\alpha k^\alpha = 0 = k_\alpha e_A^\alpha. \quad (2.3)$$

The remaining inner products

$$\sigma_{AB}(\lambda, \theta^C) \equiv g_{\alpha\beta} e_A^\alpha e_B^\beta \quad (2.4)$$

do not vanish, and we assume that they are the same on both sides of Σ :

$$[\sigma_{AB}] = 0. \quad (2.5)$$

We have introduced here the standard notation $[A] \equiv A(\mathcal{M}^+)|_\Sigma - A(\mathcal{M}^-)|_\Sigma$ for any scalar quantity A . In the first term, A is computed in \mathcal{M}^+ and evaluated on Σ ; in the second term, A is computed in \mathcal{M}^- instead. A vanishing $[A]$ means that A is continuous at the hypersurface; a nonzero $[A]$ means that A is discontinuous, and $[A]$ gives the jump of A across Σ .

Intrinsic metric

It is easy to see that σ_{AB} acts as a metric on Σ : A displacement on the hypersurface is given by $dx_\pm^\alpha = k_\pm^\alpha d\lambda + e_{\pm A}^\alpha d\theta^A$, and by virtue of Eqs. (2.3) and (2.4), we find

$$ds_\Sigma^2 = \sigma_{AB} d\theta^A d\theta^B \quad (2.6)$$

on either side of Σ . The condition (2.5) therefore ensures that the hypersurface possesses a well-defined intrinsic geometry, a central requirement of the null-shell formalism. A violation of Eq. (2.5) would indicate that the singularity of the hypersurface is of a much stronger type than can be handled by distributional techniques, and the physical interpretation of such a singular hypersurface would be much more delicate.

It should be noted that the intrinsic metric of the null hypersurface is two-dimensional. This is as it should be, as a displacement along a generator necessarily gives $ds^2 = 0$. This property, that the hypersurface metric is not merely degenerate but explicitly two-dimensional, is a definite advantage. For example, there is no difficulty in defining a metric inverse, which we denote σ^{AB} , and which is the same on both sides of the hypersurface. The inverse two-metric satisfies $\sigma^{AC} \sigma_{CB} = \delta_C^A$.

Transverse vector

The tangent vectors k_\pm^α and $e_{\pm A}^\alpha$ provide only a partial basis for the decomposition of vector fields on either side of the hypersurface. We now complete the basis by adding a transverse null vector N_\pm^α that satisfies

$$N_\alpha N^\alpha = 0, \quad N_\alpha k^\alpha = -1, \quad N_\alpha e_A^\alpha = 0. \quad (2.7)$$

These four equations give a unique specification of the transverse vector. The basis gives us the completeness relations

$$g^{\alpha\beta} = -k^\alpha N^\beta - N^\alpha k^\beta + \sigma^{AB} e_A^\alpha e_B^\beta \quad (2.8)$$

for the inverse metric on either side of Σ (in the coordinates x_\pm^α); as was noted above, σ^{AB} is the inverse of σ_{AB} . The completeness relations can be used to decompose a vector A_\pm^α in the specified basis: $A^\alpha = g^{\alpha\beta} A_\beta = (-N^\beta A_\beta) k^\alpha + (-k^\beta A_\beta) N^\alpha + (\sigma^{AB} e_B^\beta A_\beta) e_A^\alpha$, where we have omitted the \pm label for increased clarity.

Congruence of timelike geodesics

The first key aspect of our reformulation of the null-shell formalism was introduced previously: it is our choice of preferred intrinsic coordinates which, as we shall see, provides the surface stress-energy tensor with a simple description and a clear physical interpretation. The second key aspect of our reformulation is the introduction of an arbitrary congruence of timelike geodesics that intersect the null hypersurface. As we shall see, the congruence will give operational meaning to the distributional character of the material stress-energy tensor.

Each member of the congruence is thought of as the world line of a freely-moving observer that intersects the surface layer and performs measurements on it. The congruence then corresponds to a whole family of observers. We treat all such families equally and do not attempt to

introduce a preferred set of observers; the congruence is therefore completely arbitrary.

The timelike geodesics are parameterized by proper time τ , which is adjusted so that $\tau = 0$ when a geodesic intersects Σ ; thus, $\tau < 0$ in \mathcal{M}^- and $\tau > 0$ in \mathcal{M}^+ . The vector field tangent to the congruence is u_\pm^α and is normalized; a displacement along a single geodesic is given by $dx_\pm^\alpha = u_\pm^\alpha d\tau$. To ensure that the congruence is smooth at the hypersurface, we demand that u_\pm^α be “the same” on both sides of Σ . This means that $u_\alpha e_a^\alpha$, the tangential projections of the vector field, must be equal when evaluated on either side of the hypersurface:

$$[-u_\alpha k^\alpha] = 0 = [u_\alpha e_A^\alpha]. \quad (2.9)$$

If, for example, u_-^α is specified in \mathcal{M}^- , then the three conditions of Eq. (2.9) are sufficient (together with the geodesic equation) to determine the three independent components of u_+^α in \mathcal{M}^+ . We note that $-u_\alpha N^\alpha$, the transverse projection of the vector field, is allowed to be discontinuous across Σ .

The proper-time parameter on the timelike geodesics can be viewed as a scalar field $\tau(x_\pm^\alpha)$ defined in a normal convex neighbourhood of Σ : Select a point x_\pm^α off the hypersurface and locate the unique member of the congruence that links this point to Σ ; the value of the scalar field at x_\pm^α is equal to the proper-time parameter of this geodesic at that point. The hypersurface Σ can then be described by the statement $\tau(x_\pm^\alpha) = 0$, and its normal vector k_α^\pm will be proportional to the gradient of $\tau(x_\pm^\alpha)$ evaluated on Σ . It is easy to check that the appropriate expression is

$$k_\alpha = -(-k_\mu u^\mu) \frac{\partial \tau}{\partial x^\alpha} \Big|_\Sigma. \quad (2.10)$$

We recall that the factor $-k_\mu u^\mu$ in Eq. (2.10) is continuous at Σ .

Transverse curvature

We have seen that the two-dimensional induced metric σ_{AB} is necessarily the same on both sides of the hypersurface. In the presence of a surface layer, however, we can expect that the metric’s transverse derivative will be discontinuous across Σ . In Sec. III we show that information about the metric’s transverse derivative is properly encoded in the *transverse curvature* C_{ab}^\pm defined by

$$C_{ab} = -N_\alpha e_{a;\beta}^\alpha e_b^\beta = C_{ba}. \quad (2.11)$$

In the presence of a surface layer, this object (which transforms as a scalar under transformations of the coordinates x_\pm^α) is discontinuous across Σ : $[C_{ab}] \neq 0$. The symmetry of C_{ab}^\pm follows from the identity $e_{a;\beta}^\alpha e_b^\beta = e_{b;\beta}^\alpha e_a^\beta$, which states that each basis vector e_a^α is Lie transported along any other basis vector. Note that C_{ab} is the same object that was denoted \mathcal{K}_{ab} in the original work by Barrabès and Israel [10].

Surface stress-energy tensor

The jump in the transverse curvature is directly related to the stress-energy tensor of the surface layer. To display this relationship we introduce the surface quantities

$$\mu = -\frac{1}{8\pi} \sigma^{AB} [C_{AB}], \quad (2.12)$$

$$j^A = \frac{1}{8\pi} \sigma^{AB} [C_{\lambda B}], \quad (2.13)$$

$$p = -\frac{1}{8\pi} [C_{\lambda\lambda}]. \quad (2.14)$$

We prove in Sec. III that the layer’s surface stress-energy tensor is given on either side of the hypersurface by

$$S^{\alpha\beta} = \mu k^\alpha k^\beta + j^A (k^\alpha e_A^\beta + e_A^\alpha k^\beta) + p \sigma^{AB} e_A^\alpha e_B^\beta. \quad (2.15)$$

In this expression, the first term represents a flow of matter along the null generators of the hypersurface, and the surface quantity μ therefore represents a mass density. The second term represents a flow of matter in the directions transverse to the generators, and the surface quantities j^A therefore represent a current density. Finally, the surface quantity p clearly represents an isotropic pressure. The surface layer can therefore be characterized by four quantities: a density μ , a current j^A , and an isotropic pressure p . This provides the surface stress-energy tensor with a transparent physical interpretation. As was noted before, this clarity of interpretation comes from employing the null generators to define a preferred flow on the hypersurface, to which the flow of matter can be compared.

We show in Sec. III that the complete stress-energy tensor of the surface layer is given by

$$T_\Sigma^{\alpha\beta} = (-k_\mu u^\mu)^{-1} S^{\alpha\beta} \delta(\tau). \quad (2.16)$$

In this expression, the factor $(-k_\mu u^\mu)^{-1}$ is continuous at Σ , and (as was discussed previously) the vector field u_\pm^α is tangent to an arbitrary congruence of timelike geodesics parameterized by τ . The singular character of the stress-energy tensor is revealed by the Dirac δ -function, and the congruence of timelike geodesics was introduced for the specific purpose of making the argument of the δ -function meaningful. Indeed, the fact that it is τ — proper time as measured by a family of freely-moving observers intersecting the hypersurface — and nothing else confers immediate operational meaning to the prefactor $(-k_\mu u^\mu)^{-1} S_\pm^{\alpha\beta}$. This choice of argument therefore clarifies the physical meaning of the surface stress-energy tensor.

It is important to note that the presence of the factor $(-k_\mu u^\mu)^{-1}$ in front of $S_\pm^{\alpha\beta}$ in Eq. (2.16) implies that μ , j^A , and p are not truly the surface quantities that would be measured by the freely-moving observers. The physically-measured surface quantities are instead given by

$$\mu_{\text{pm}} = (-k_\mu u^\mu)^{-1} \mu, \quad (2.17)$$

$$j_{\text{pm}}^A = (-k_\mu u^\mu)^{-1} j^A, \quad (2.18)$$

$$p_{\text{pm}} = (-k_\mu u^\mu)^{-1} p. \quad (2.19)$$

It is remarkable that the ambiguity associated with the arbitrary choice of congruence is limited to a single multiplicative factor. The “bare” quantities μ , j^A , and p are independent of this choice, and it is often more convenient to work in terms of those.

User manual

Our reformulation of the Barrabès-Israel null-shell formalism is now complete. The formalism can be used as follows. First, locate the null hypersurface Σ in \mathcal{M}^\pm and construct the intrinsic coordinates $y^a = (\lambda, \theta^A)$; this step is the only nonalgorithmic operation involved in the recipe. Second, construct the tangent vectors k_\pm^α , $e_{\pm A}^\alpha$ and complete the basis by computing the transverse null vector N_α^\pm . Third, compute the intrinsic metric σ_{AB} and verify that it is the same on both sides of the hypersurface. Fourth, compute the transverse curvature C_{ab}^\pm and its jump across Σ . Fifth, and finally, compute the surface quantities μ , j^A , and p .

III. DERIVATIONS

In this section we provide derivations of the main results presented in Sec. II, namely Eqs. (2.12)–(2.16). Following the Appendix of the Barrabès-Israel paper [10], we introduce a suitable coordinate system x^μ in a coordinate neighbourhood that includes the null hypersurface Σ and extends into both regions \mathcal{M}^\pm . These coordinates, which are distinct from the systems x_\pm^μ encountered before, will be used for the derivations, but as we saw in Sec. II, the final formulation of the null-shell formalism will not refer to them.

Singular part of the Riemann tensor

In the domain of our suitable coordinate system we write the metric as a distribution-valued tensor:

$$g_{\alpha\beta} = g_{\alpha\beta}^+ \Theta(\tau) + g_{\alpha\beta}^- \Theta(-\tau), \quad (3.1)$$

where $\Theta(\tau)$ is the Heaviside step function and $g_{\alpha\beta}^\pm(x^\mu)$ is the metric in \mathcal{M}^\pm . To ensure that the metric is well defined at $\tau = 0$ we assume that in the coordinates x^μ , $[g_{\alpha\beta}] = 0$; Eq. (2.5) is compatible with this requirement. We also impose $[k^\alpha] = [e_A^\alpha] = [N^\alpha] = [u^\alpha] = 0$, and these continuity statements provide a precise meaning to the phrase “suitable coordinate system.”

The metric of Eq. (3.1) generally gives rise to a singular Riemann tensor, even when continuity of $g_{\alpha\beta}^\pm$ at $\tau = 0$ is enforced; the reason is that the *transverse derivative* of

the metric might be discontinuous at the hypersurface. To compute the Riemann tensor we use Eq. (2.10) to relate the gradient of τ to the null vector k^α . The result is

$$R_{\beta\gamma\delta}^\alpha = R_+{}^\alpha{}_{\beta\gamma\delta} \Theta(\tau) + R_-{}^\alpha{}_{\beta\gamma\delta} \Theta(-\tau) + R_\Sigma{}^\alpha{}_{\beta\gamma\delta}, \quad (3.2)$$

where $R_\pm{}^\alpha{}_{\beta\gamma\delta}$ denotes the Riemann tensor in \mathcal{M}^\pm constructed from $g_{\alpha\beta}^\pm$, and

$$R_\Sigma{}^\alpha{}_{\beta\gamma\delta} = -(-k_\mu u^\mu)^{-1} ([\Gamma_{\beta\delta}^\alpha] k_\gamma - [\Gamma_{\beta\gamma}^\alpha] k_\delta) \delta(\tau), \quad (3.3)$$

is the singular part of the Riemann tensor. Here, $[\Gamma_{\beta\gamma}^\alpha]$ denotes the jump in the affine connection across Σ .

We must now characterize the discontinuous behaviour of $g_{\alpha\beta,\gamma}$. The condition $[g_{\alpha\beta}] = 0$ guarantees that the tangential derivatives of the metric are continuous: $[g_{\alpha\beta,\gamma}] k^\gamma = 0 = [g_{\alpha\beta,\gamma}] e_C^\gamma$. The only possible discontinuity is therefore in $g_{\alpha\beta,\gamma} N^\gamma$, the transverse derivative of the metric. In view of Eq. (2.7) we conclude that there exists a tensor field $\gamma_{\alpha\beta}$ such that

$$[g_{\alpha\beta,\gamma}] = -\gamma_{\alpha\beta} k_\gamma. \quad (3.4)$$

This tensor is given explicitly by $\gamma_{\alpha\beta} = [g_{\alpha\beta,\gamma}] N^\gamma$ and it follows that

$$[\Gamma_{\beta\gamma}^\alpha] = -\frac{1}{2} (\gamma_{\beta\gamma}^\alpha k_\gamma + \gamma_{\gamma\beta}^\alpha k_\beta - \gamma_{\beta\gamma}^\alpha k^\alpha). \quad (3.5)$$

Substituting this into Eq. (3.3) gives

$$R_\Sigma{}^\alpha{}_{\beta\gamma\delta} = \frac{1}{2} (-k_\mu u^\mu)^{-1} (\gamma_{\delta\beta}^\alpha k_\gamma k_\gamma - \gamma_{\beta\delta}^\alpha k^\alpha k_\gamma - \gamma_{\gamma\beta}^\alpha k_\delta k_\delta + \gamma_{\beta\gamma}^\alpha k^\alpha k_\delta) \delta(\tau), \quad (3.6)$$

and we see that k^α and $\gamma_{\alpha\beta}$ give a complete characterization of the singular part of the Riemann tensor.

Surface stress-energy tensor

From Eq. (3.6) we form the singular part of the Einstein tensor, and the Einstein field equations give us the singular part of the stress-energy tensor:

$$T_\Sigma^{\alpha\beta} = (-k_\mu u^\mu)^{-1} S^{\alpha\beta} \delta(\tau), \quad (3.7)$$

where

$$S^{\alpha\beta} = \frac{1}{16\pi} (k^\alpha \gamma_{\mu}^\beta k^\mu + k^\beta \gamma_{\mu}^\alpha k^\mu - \gamma_{\mu}^\mu k^\alpha k^\beta - \gamma_{\mu\nu} k^\mu k^\nu g^{\alpha\beta}) \quad (3.8)$$

is the surface stress-energy tensor of the null shell, up to a factor $(-k_\mu u^\mu)$ that depends on the choice of observers making measurements on the shell.

The expression for the surface stress-energy tensor can be simplified if we decompose it into the basis $(k^\alpha, N^\alpha, e_A^\alpha)$. For this purpose we introduce the projections

$$\gamma_A \equiv \gamma_{\alpha\beta} e_A^\alpha k^\beta, \quad \gamma_{AB} \equiv \gamma_{\alpha\beta} e_A^\alpha e_B^\beta, \quad (3.9)$$

and we use the completeness relations of Eq. (2.8) to find that the vector $\gamma^\alpha_\mu k^\mu$ admits the decomposition

$$\gamma^\alpha_\mu k^\mu = \frac{1}{2}(\gamma^\mu_\mu - \sigma^{AB}\gamma_{AB})k^\alpha - (\gamma_{\mu\nu}k^\mu k^\nu)N^\alpha + (\sigma^{AB}\gamma_B) e^\alpha_A. \quad (3.10)$$

Substituting this into Eq. (3.8) and involving once more the completeness relations, we arrive at our final expression for the surface stress-energy tensor:

$$S^{\alpha\beta} = \mu k^\alpha k^\beta + j^A (k^\alpha e^\beta_A + e^\alpha_A k^\beta) + p \sigma^{AB} e^\alpha_A e^\beta_B, \quad (3.11)$$

where

$$\mu \equiv -\frac{1}{16\pi} (\sigma^{AB}\gamma_{AB}) \quad (3.12)$$

is the surface density,

$$j^A \equiv \frac{1}{16\pi} (\sigma^{AB}\gamma_B) \quad (3.13)$$

is the surface current, and

$$p \equiv -\frac{1}{16\pi} (\gamma_{\alpha\beta} k^\alpha k^\beta) \quad (3.14)$$

is the isotropic surface pressure.

The surface stress-energy tensor of Eq. (3.11) is expressed in the suitable coordinates x^μ . This is, however, a tensorial equation involving vectors (k^α and e^α_A) and scalars (μ , j^A , and p). This equation can therefore be expressed in any coordinate system. In particular, when viewed from \mathcal{M}^\pm , the surface stress-energy tensor can be expressed in the original coordinates x^\pm_\pm . This means that Eq. (3.7) is formally identical to Eq. (2.16), while Eq. (3.11) is identical to Eq. (2.15).

Intrinsic formulation

The surface quantities of Eqs. (3.12)–(3.14) are constructed from the tensor $\gamma_{\alpha\beta}$ which is defined only in the suitable coordinates x^μ . As it might be grossly impractical to construct such a coordinate system, it is important to find an independent way of obtaining these quantities. It is at this stage that we introduce the transverse curvature C_{ab} defined by Eq. (2.11). Recalling from Eq. (2.7) that $N_\alpha e^\alpha_a$ are constants (either zero or minus one), we express the transverse curvature as

$$C_{ab} = N_{\alpha;\beta} e^\alpha_a e^\beta_b \quad (3.15)$$

and we seek to relate its jump to the surface quantities. In the next paragraph we will use the suitable coordinates x^μ to establish the general validity of the scalar relations (2.12)–(2.14). This will bring our derivations to a close.

In the suitable coordinates, the jump in the transverse curvature is given by

$$\begin{aligned} [C_{ab}] &= [N_{\alpha;\beta}] e^\alpha_a e^\beta_b = -[\Gamma^\gamma_{\alpha\beta}] N_\gamma e^\alpha_a e^\beta_b \\ &= \frac{1}{2} \gamma_{\alpha\beta} e^\alpha_a e^\beta_b, \end{aligned} \quad (3.16)$$

where we have used Eq. (3.5) and the fact that k^α is orthogonal to e^α_a . We therefore have $[C_{\lambda\lambda}] = \frac{1}{2}\gamma_{\alpha\beta} k^\alpha k^\beta$, $[C_{A\lambda}] = \frac{1}{2}\gamma_{\alpha\beta} e^\alpha_A k^\beta \equiv \frac{1}{2}\gamma_A$, and $[C_{AB}] = \frac{1}{2}\gamma_{\alpha\beta} e^\alpha_A e^\beta_B \equiv \frac{1}{2}\gamma_{AB}$, where we have involved Eq. (3.9). Finally, using Eqs. (3.12)–(3.14) we find that the surface quantities can indeed be expressed as in Eqs. (2.12)–(2.14).

We have established that the shell's surface quantities can all be computed directly in terms of the induced metric σ_{AB} and the jump of the transverse curvature C_{ab} across Σ . This confirms that the suitable coordinates x^μ are not needed in practical applications of the null-shell formalism.

IV. PARAMETERIZATION OF THE NULL GENERATORS

In this section we discuss two technical issues related to the parameterization of the null generators of the singular hypersurface. In the first subsection we justify a statement made earlier, that in general λ cannot be an affine parameter on both sides of the hypersurface; our presentation here is based directly on Sec. III of the Barrabès-Israel paper [10], but for completeness we adapt their discussion to the notation of this paper. In the second subsection we examine the transformation properties of the surface quantities under a reparameterization of the null generators.

Affine parameterization of the null generators

Whether or not λ is an affine parameter can be decided by computing κ_\pm , the “acceleration” of the null vector k^α_\pm . This is defined on either side of the hypersurface by

$$k^\alpha_{;\beta} k^\beta = \kappa k^\alpha, \quad (4.1)$$

and λ will be an affine parameter on the \mathcal{M}^\pm side of Σ if $\kappa_\pm = 0$. According to Eq. (2.7), $\kappa = -N_\alpha k^\alpha_{;\beta} k^\beta = -N_\alpha e^\alpha_{\lambda;\beta} e^\beta_\lambda = C_{\lambda\lambda}$, where we have also used Eqs. (2.2) and (2.11). Equation (2.14) then relates the discontinuity in the acceleration to the surface pressure:

$$[\kappa] = -8\pi p. \quad (4.2)$$

We conclude that λ can be an affine parameter on both sides of Σ only when the surface layer has a vanishing surface pressure. When $p \neq 0$, on the other hand, λ can be chosen to be an affine parameter on *one* side of the hypersurface, but it will not be an affine parameter on the other side.

Additional insight into this matter can be gained from Raychaudhuri's equation,

$$\frac{d\theta}{d\lambda} + \frac{1}{2}\theta^2 + \sigma^{\alpha\beta}\sigma_{\alpha\beta} = \kappa\theta - 8\pi T_{\alpha\beta} k^\alpha k^\beta, \quad (4.3)$$

which describes the evolution of the congruence of null generators [24, 25]; the equation holds on either side of

the hypersurface. Here, θ and $\sigma_{\alpha\beta}$ are the expansion and shear of the congruence, respectively, and we have set the vorticity to zero because the congruence is hypersurface orthogonal. Because it depends only on the intrinsic geometry of the hypersurface, the left-hand side of Raychaudhuri's equation is necessarily continuous across the shell. Continuity of the right-hand side therefore implies

$$[\kappa]\theta = 8\pi[T_{\alpha\beta}k^\alpha k^\beta]. \quad (4.4)$$

This relation shows that $[\kappa] \neq 0$ (and therefore $p \neq 0$) whenever the component $T_{\alpha\beta}k^\alpha k^\beta$ of the stress-energy tensor is discontinuous across the shell. We conclude that λ cannot be an affine parameter on both sides of Σ when $[T_{\alpha\beta}k^\alpha k^\beta] \neq 0$. (Notice that this conclusion breaks down when $\theta = 0$, that is, when the shell is stationary.)

Recalling that the expansion θ is equal to the fractional rate of change of the congruence's cross-sectional area, we find that with the help of Eq. (4.2), Eq. (4.4) can be expressed as $-p\theta = [T_{\alpha\beta}k^\alpha k^\beta]$, or

$$-p \frac{d}{d\lambda} dS = [T_{\alpha\beta}k^\alpha k^\beta] dS, \quad (4.5)$$

where dS is an element of area transverse to the generators. This equation has a straightforward interpretation: The left-hand side represents the work done on the shell as it expands or contracts, while the right-hand side is the energy absorbed by the shell from its surroundings; Eq. (4.5) therefore states that all of the absorbed energy goes into work.

Reparameterization of the null generators

We now investigate how a change of parameterization affects the surface density μ , surface current j^A , and surface pressure p of the null shell. Because each generator can be reparameterized independently of any other generator, we must consider transformations of the form

$$\lambda \rightarrow \bar{\lambda}(\lambda, \theta^A). \quad (4.6)$$

The question before us is: How do the surface quantities change under such a transformation?

To answer this we need to work out how the transformation of Eq. (4.6) affects the vectors k^α , e_A^α , and N^α . We first note that the differential form of Eq. (4.6) is

$$d\bar{\lambda} = e^\beta d\lambda + c_A d\theta^A, \quad (4.7)$$

where

$$e^\beta \equiv \left(\frac{\partial \bar{\lambda}}{\partial \lambda} \right)_{\theta^A}, \quad c_A \equiv \left(\frac{\partial \bar{\lambda}}{\partial \theta^A} \right)_\lambda; \quad (4.8)$$

both e^β and c_A depend on $y^a = (\lambda, \theta^A)$, but because they depend on the intrinsic coordinates only, we have that $[e^\beta] = 0 = [c_A]$. A displacement within the hypersurface can then be described either by

$$dx^\alpha = k^\alpha d\lambda + e_A^\alpha d\theta^A, \quad (4.9)$$

where $k^\alpha = (\partial x^\alpha / \partial \lambda)_{\theta^A}$ and $e_A^\alpha = (\partial x^\alpha / \partial \theta^A)_\lambda$, or by

$$dx^\alpha = \bar{k}^\alpha d\bar{\lambda} + \bar{e}_A^\alpha d\theta^A, \quad (4.10)$$

where $\bar{k}^\alpha = (\partial x^\alpha / \partial \bar{\lambda})_{\theta^A}$ and $\bar{e}_A^\alpha = (\partial x^\alpha / \partial \theta^A)_{\bar{\lambda}}$; these relations hold on either side of Σ , in the relevant coordinate system x_\pm^α . Using Eq. (4.7), it is easy to see that the tangent vectors transform as

$$\bar{k}^\alpha = e^{-\beta} k^\alpha, \quad \bar{e}_A^\alpha = e_A^\alpha - c_A e^{-\beta} k^\alpha \quad (4.11)$$

under the reparameterization of Eq. (4.6). It may be checked that the new basis vectors satisfy the orthogonality relations of Eq. (2.3), and that the induced metric is invariant under this transformation: $\bar{\sigma}_{AB} \equiv g_{\alpha\beta} \bar{e}_A^\alpha \bar{e}_B^\beta = g_{\alpha\beta} e_A^\alpha e_B^\beta \equiv \sigma_{AB}$. To preserve the relations (2.7) we let the new transverse null vector be

$$\bar{N}^\alpha = e^\beta N^\alpha + \frac{1}{2} c^A c_A e^{-\beta} k^\alpha - c^A e_A^\alpha, \quad (4.12)$$

where $c^A = \sigma^{AB} c_B$. This ensures that the completeness relations of Eq. (2.8) take the same form in the new basis.

It is a straightforward (but slightly tedious) task to compute how the transverse curvature C_{ab} changes under a reparameterization of the generators, and to then compute how the surface quantities transform. We find

$$\bar{\mu} = e^\beta \mu + 2c_A j^A + c^A c_A e^{-\beta} p, \quad (4.13)$$

$$\bar{j}^A = j^A + c^A e^{-\beta} p, \quad (4.14)$$

$$\bar{p} = e^{-\beta} p. \quad (4.15)$$

These transformations, together with Eq. (4.11), imply that the surface stress-energy tensor becomes $\bar{S}^{\alpha\beta} = e^{-\beta} S^{\alpha\beta}$. We also have $(-\bar{k}_\mu u^\mu)^{-1} = e^\beta (-k_\mu u^\mu)$, and these results imply that the combination $(-k_\mu u^\mu)^{-1} S^{\alpha\beta}$ is *invariant* under the reparameterization. This, finally, establishes the invariance of $T_\Sigma^{\alpha\beta}$.

As a final remark, we note that under the reparameterization of Eq. (4.6), the physically-measured surface quantities of Eqs. (2.17)–(2.19) transform as

$$\bar{\mu}_{\text{pm}} = e^{2\beta} \mu_{\text{pm}} + 2c_A e^\beta j_{\text{pm}}^A + c^A c_A p_{\text{pm}}, \quad (4.16)$$

$$\bar{j}_{\text{pm}}^A = e^\beta j_{\text{pm}}^A + c^A p_{\text{pm}}, \quad (4.17)$$

$$\bar{p}_{\text{pm}} = p_{\text{pm}}; \quad (4.18)$$

we see in particular that the physically-measured surface pressure is an invariant.

V. APPLICATIONS

In this section we consider four simple applications of the null-shell formalism. Those are designed to illustrate the basic methods involved and highlight some aspects of the physics of null surface layers.

Imploding spherical shell

For our first application of the null-shell formalism, we look at the gravitational collapse of a thin spherical shell. We imagine that the collapse proceeds at the speed of light, so that the world surface of the shell coincides with a null hypersurface Σ . We take spacetime to be flat inside the shell (in \mathcal{M}^-), and write the metric there as

$$ds_-^2 = -dt_-^2 + dr^2 + r^2 d\Omega^2, \quad (5.1)$$

in terms of spatial coordinates (r, θ, ϕ) and a time coordinate t_- . The metric outside the shell (in \mathcal{M}^+) is the Schwarzschild solution,

$$ds_+^2 = -f dt_+^2 + f^{-1} dr^2 + r^2 d\Omega^2, \quad (5.2)$$

expressed in the same spatial coordinates but in terms of a distinct time t_+ ; we have $f = 1 - 2M/r$ and M designates the gravitational mass of the imploding shell.

As seen from \mathcal{M}^- , the null hypersurface is described by the equation $t_- + r \equiv v_- = \text{constant}$, which implies that the induced metric on Σ is given by $ds_\Sigma^2 = r^2 d\Omega^2$. As seen from \mathcal{M}^+ , on the other hand, the hypersurface is described by $t_+ + r^*(r) \equiv v_+ = \text{constant}$, where $r^*(r) = \int f^{-1} dr = r + 2M \ln(r/2M - 1)$, and this gives rise to the same induced metric. From these considerations we conclude that it was permissible to express the metrics of \mathcal{M}^\pm in terms of the same spatial coordinates (r, θ, ϕ) , but that t_+ cannot be equal to t_- . The shell's intrinsic metric is

$$\sigma_{AB} d\theta^A d\theta^B = \lambda^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (5.3)$$

where we have set $\theta^A = (\theta, \phi)$ and identified $-r$ with the parameter λ on the null generators of the hypersurface. We shall see that here, λ is an affine parameter on both sides of Σ .

As seen from \mathcal{M}^- , the parametric equations $x_-^\alpha = x_-^\alpha(\lambda, \theta^A)$ that describe the hypersurface have the explicit form $t_- = v_- + \lambda$, $r = -\lambda$, $\theta = \theta$, and $\phi = \phi$. These give us the tangent vectors $k^\alpha \partial_\alpha = \partial_t - \partial_r$, $e_\theta^\alpha \partial_\alpha = \partial_\theta$, and $e_\phi^\alpha \partial_\alpha = \partial_\phi$, and the basis is completed by the transverse vector $N_\alpha dx^\alpha = -\frac{1}{2}(dt - dr)$. From all this and Eq. (2.11) we find that the nonvanishing components of the transverse curvature are

$$C_{AB}^- = \frac{1}{2r} \sigma_{AB}. \quad (5.4)$$

The fact that $C_{\lambda\lambda}^- = 0$ confirms that $\lambda \equiv -r$ is an affine parameter on the \mathcal{M}^- side of Σ .

As seen from \mathcal{M}^+ , the parametric equations are $t_+ = v_+ - r^*(-\lambda)$, $r = -\lambda$, $\theta = \theta$, and $\phi = \phi$. The basis vectors are $k^\alpha \partial_\alpha = f^{-1} \partial_t - \partial_r$, $e_\theta^\alpha \partial_\alpha = \partial_\theta$, $e_\phi^\alpha \partial_\alpha = \partial_\phi$, and $N_\alpha dx^\alpha = -\frac{1}{2}(f dt - dr)$. The nonvanishing components of the transverse curvature are now

$$C_{AB}^+ = \frac{f}{2r} \sigma_{AB}. \quad (5.5)$$

The fact that $C_{\lambda\lambda}^+ = 0$ confirms that $\lambda \equiv -r$ is an affine parameter on the \mathcal{M}^+ side of Σ ; λ is therefore an affine parameter on both sides.

The angular components of the transverse curvature are discontinuous across the shell: $[C_{AB}] = -(M/r^2) \sigma_{AB}$. According to Eqs. (2.12)–(2.14), this means that the shell has a vanishing surface current j^A and a vanishing surface pressure p , but that its surface density is

$$\mu = \frac{M}{4\pi r^2}. \quad (5.6)$$

We have therefore obtained the very sensible result that the surface density of an imploding null shell is equal to its gravitational mass divided by its (ever decreasing) surface area. Notice that $\mu_{\text{pm}} = \mu$ for observers at rest in \mathcal{M}^- . Because of the gravitational action of the null shell, however, these observers do not remain at rest after crossing over to the \mathcal{M}^+ side: A simple calculation, based on Eq. (2.9), reveals that an observer at rest before crossing the shell will move according to $dr/d\tau = -(\gamma^2 - f)^{1/2}$ after crossing the shell; the constant γ varies from observer to observer, and is related by $\gamma = 1 - M/r_\Sigma$ to the radius r_Σ at which a given observer crosses the hypersurface.

Spherical null shells

In our second application of the null-shell formalism, we extend the results of the preceding subsection to a situation in which the surface layer is immersed in a general spherically-symmetric spacetime. Our results here are identical to those first presented by Barrabès and Israel in Sec. IV of their paper [10]. We adapt their discussion to the notation of this paper for completeness, as spherically-symmetric situations are of extreme practical importance. We assume that the surface layer is either expanding or contracting, and we exclude stationary hypersurfaces from our considerations. For a discussion of the interesting special case of horizon-straggling shells, we refer the reader to the original Barrabès-Israel work.

The metric in \mathcal{M}^\pm is expressed in the general form

$$ds^2 = -e^\psi dw (f e^\psi dw + 2\zeta dr) + r^2 d\Omega^2, \quad (5.7)$$

where ψ_\pm and f_\pm are two arbitrary functions of the coordinates w_\pm and r ; we also have the parameter $\zeta = \pm 1$ whose meaning will be explained shortly. (Notice that the same r is used as a coordinate in both regions.) It is convenient to introduce the mass functions $m_\pm(w_\pm, r)$ defined by the relation

$$f = 1 - 2m/r. \quad (5.8)$$

The hypersurfaces $w_\pm = \text{constant}$ are null. They are expanding (r increases along the generators) if $\zeta = +1$, and w_\pm (which normally would be denoted u_\pm) is then properly interpreted as a retarded-time coordinate. On the other hand, the null hypersurfaces are contracting

(r decreases along the generators) if $\zeta = -1$, and w_{\pm} (which normally would be denoted v_{\pm}) is then properly interpreted as an advanced-time coordinate.

The boundary between \mathcal{M}^- and \mathcal{M}^+ is taken to be a particular hypersurface $w_{\pm} = \text{constant}$. According to Eq. (5.7), its intrinsic metric is given by

$$\sigma_{AB} d\theta^A d\theta^B = r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (5.9)$$

which is the same on both sides of Σ . We have identified θ^A with the two angles (θ, ϕ) , and we take $\lambda \equiv \zeta r$ to be the parameter on the null generators. Notice that λ always increases along the generators.

The parametric equations that describe the hypersurface are $w_{\pm} = \text{constant}$, $r = \zeta \lambda$, $\theta = \theta$, and $\phi = \phi$. The tangent vectors are then

$$k^{\alpha} \partial_{\alpha} = \zeta \partial_r, \quad e_{\theta}^{\alpha} \partial_{\alpha} = \partial_{\theta}, \quad e_{\phi}^{\alpha} \partial_{\alpha} = \partial_{\phi} \quad (5.10)$$

on either side of the hypersurface. The basis is completed by the transverse null vector

$$N_{\alpha} dx^{\alpha} = -\frac{1}{2} (f e^{\psi} dw + 2\zeta dr). \quad (5.11)$$

We may now use Eq. (2.11) to compute the transverse curvature. The nonvanishing components are

$$C_{\lambda\lambda} \equiv \kappa = \zeta \frac{\partial \psi}{\partial r}, \quad C_{AB} = -\zeta \frac{f}{2r} \sigma_{AB}. \quad (5.12)$$

We recall from Sec. IV that κ_{\pm} is the “acceleration” of the null vector k_{\pm}^{α} ; from the first equation we therefore learn that λ is an affine parameter on the \mathcal{M}^{\pm} side of Σ when $\partial \psi_{\pm} / \partial r = 0$.

According to Eqs. (2.12)–(2.14) and (5.12), the only nonvanishing surface quantities are μ and p . Using Eq. (5.8) we find that the mass density is given by

$$\mu = (-\zeta) \frac{[m]}{4\pi r^2}; \quad (5.13)$$

this is the difference in the mass function across the shell, $[m](r)$, divided by the shell’s surface area, $4\pi r^2$. Assuming that μ is positive, we see that $[m] < 0$ if the hypersurface is expanding — the shell removes energy from the centre — whereas $[m] > 0$ if the hypersurface is contracting — the shell then brings energy to the centre. The shell’s surface pressure is given by

$$p = (-\zeta) \frac{1}{8\pi} \left[\frac{\partial \psi}{\partial r} \right]. \quad (5.14)$$

This result can be related to the first law of thermodynamics if we note that $[\partial \psi / \partial r] = 4\pi r [T_{\alpha\beta} k^{\alpha} k^{\beta}]$ according to the Einstein field equations, and that the expansion of the null generators is given by $\theta = k^{\alpha}_{;\alpha} - \kappa = 2\xi/r$. Equation (5.13) is then seen to be equivalent to the statement $p\theta = -[T_{\alpha\beta} k^{\alpha} k^{\beta}]$ previously displayed in Eq. (4.5).

Accreting black hole

In our third application of the null-shell formalism, we consider a nonrotating black hole of mass $(M - m)$ which suddenly acquires additional material of mass m and angular momentum $J \equiv aM \ll M^2$. We describe the accreting material by a singular stress-energy tensor supported on a null hypersurface Σ .

The spacetime in the future of Σ — in \mathcal{M}^+ — is that of a slowly rotating black hole of mass M and (small) angular momentum aM . We write the metric in \mathcal{M}^+ as

$$ds_+^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 - \frac{4Ma}{r} \sin^2 \theta dt d\phi, \quad (5.15)$$

where $f = 1 - 2M/r$; this is the slow-rotation limit of the Kerr metric, and throughout this subsection we will work consistently to first order in the small parameter a .

As seen from \mathcal{M}^+ , the null hypersurface is described by $v \equiv t + r^* = 0$, where $r^* = \int f^{-1} dr = r + 2M \ln(r/2M - 1)$. In the slow-rotation limit, every surface $v = \text{constant}$ is null, and it follows that the vector $k^{\alpha} = g^{\alpha\beta} (-\partial_{\beta} v)$ is normal to Σ and tangent to its null generators. We have

$$k^{\alpha} \partial_{\alpha} = \frac{1}{f} \partial_t - \partial_r + \frac{2Ma}{r^3 f} \partial_{\phi}, \quad (5.16)$$

and from this expression we deduce four important properties of the generators. First, the generators are affinely parameterized by $\lambda \equiv -r$. Second, as measured by inertial observers at infinity, the generators move with an (ever increasing) angular velocity

$$\frac{d\phi}{dt} \equiv \Omega_{\text{generators}} = \frac{2Ma}{r^3}. \quad (5.17)$$

Third, θ is constant on each generator. And fourth, integration of $d\phi/(-dr) = 2Ma/(r^3 f)$ reveals that

$$\psi \equiv \phi + \frac{a}{r} \left(1 + \frac{r}{2M} \ln f \right) \quad (5.18)$$

also is constant on the generators.

We shall use $y^a = (\lambda \equiv -r, \theta, \psi)$ as coordinates on Σ , as these are well adapted to the generators. Remembering that $dt = -dr/f$ and $d\phi = d\psi - (2Ma/r^3 f) dr$ on Σ , we find that the induced metric is

$$\sigma_{AB} d\theta^A d\theta^B = r^2 (d\theta^2 + \sin^2 \theta d\psi^2), \quad (5.19)$$

and we see that the hypersurface is intrinsically spherical.

The parametric description of Σ , as seen from \mathcal{M}^+ , is $x^{\alpha}(-r, \theta, \psi)$, and from this we form the tangent vectors $e_{\lambda}^{\alpha} = k^{\alpha}$, $e_{\theta}^{\alpha} = \delta_{\theta}^{\alpha}$, and $e_{\psi}^{\alpha} = \delta_{\psi}^{\alpha}$. The basis is completed by the transverse null vector $N_{\alpha} dx^{\alpha} = \frac{1}{2}(-f dt + dr)$. From Eq. (2.11) we obtain

$$C_{\lambda\psi}^+ = \frac{3Ma}{r^2} \sin^2 \theta, \quad C_{AB}^+ = \frac{f}{2r} \sigma_{AB} \quad (5.20)$$

for the nonvanishing components of the transverse curvature.

The spacetime in the past of Σ — in \mathcal{M}^- — is that of a nonrotating black hole of mass $(M - m)$. Here we write the metric as

$$ds_-^2 = -F d\bar{t}^2 + F^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\psi^2), \quad (5.21)$$

in terms of a distinct time coordinate \bar{t} and the angles θ and ψ ; we also have $F \equiv 1 - 2(M - m)/r$. This choice of angular coordinates reflects the fact that zero-angular-momentum observers in \mathcal{M}^- corotate with the shell's null generators; this is a manifestation of the dragging of inertial frames. As we shall see presently, this choice of coordinates is dictated by continuity of the induced metric at Σ .

The mathematical description of the hypersurface, as seen from \mathcal{M}^- , is identical to its external description provided that we make the substitutions $t \rightarrow \bar{t}$, $\phi \rightarrow \psi$, $M \rightarrow M - m$, and $a \rightarrow 0$. According to this, the induced metric on Σ is still given by $ds_\Sigma^2 = r^2(d\theta^2 + \sin^2\theta d\psi^2)$, as required. The basis vectors are now $k^\alpha \partial_\alpha = F^{-1} \partial_{\bar{t}} - \partial_r$, $e_\theta^\alpha \partial_\alpha = \partial_\theta$, $e_\psi^\alpha \partial_\alpha = \partial_\psi$, and $N_\alpha dx^\alpha = \frac{1}{2}(-F d\bar{t} + dr)$. This gives us

$$C_{AB}^- = \frac{F}{2r} \sigma_{AB} \quad (5.22)$$

for the nonvanishing components of the transverse curvature.

The transverse curvature is discontinuous at Σ , and Eqs. (2.12)–(2.14) allow us to compute the shell's surface quantities. Because the generators are affinely parameterized by $-r$ on both sides of the hypersurface, we have that $p = 0$ — the shell has a vanishing surface pressure. On the other hand, its surface density is given by

$$\mu = \frac{m}{4\pi r^2}, \quad (5.23)$$

the ratio of the shell's gravitational mass m to its (ever decreasing) surface area $4\pi r^2$. Thus far our results are virtually identical to those obtained in the first subsection. What is new in this context is the presence of a surface current j^A , whose nonvanishing component is

$$j^\psi = \frac{3Ma}{8\pi r^4}. \quad (5.24)$$

This comes from the shell's rotation, and the fact that the situation is not entirely spherically symmetric.

To better understand the physical significance of the surface current, we express the shell's surface stress-energy tensor,

$$S^{\alpha\beta} = \mu k^\alpha k^\beta + j^\psi (k^\alpha e_\psi^\beta + e_\psi^\alpha k^\beta), \quad (5.25)$$

in terms of the vector $\ell^\alpha \equiv k^\alpha + (j^\psi/\mu) e_\psi^\alpha$. This vector is null (in the slow-rotation limit) and has the components

$$\ell^\alpha \partial_\alpha = \frac{1}{f} \partial_t - \partial_r + \frac{1}{f} \Omega_{\text{fluid}} \partial_\phi \quad (5.26)$$

in the coordinates $x^\alpha = (t, r, \theta, \phi)$ used in \mathcal{M}^+ ; we have set

$$\Omega_{\text{fluid}} \equiv \frac{2Ma}{r^3} + \frac{3Ma}{2mr} f. \quad (5.27)$$

The shell's surface stress-energy tensor is now given by the simple expression

$$S^{\alpha\beta} = \mu \ell^\alpha \ell^\beta, \quad (5.28)$$

which corresponds to a pressureless fluid of density μ moving with a four-velocity ℓ^α . We see that the fluid is moving along null curves (not geodesics!) that do not coincide with the shell's null generators. This motion across generators is created by a mismatch between Ω_{fluid} , the fluid's angular velocity, and $\Omega_{\text{generators}}$, the angular velocity of the generators. The discrepancy is directly related to j^A :

$$\Omega_{\text{relative}} \equiv \Omega_{\text{fluid}} - \Omega_{\text{generators}} = \frac{j^\psi}{\mu} = \frac{3Ma}{2mr} f. \quad (5.29)$$

Notice that the fluid rotates faster than the generators, which share their angular velocity with zero-angular-momentum observers within \mathcal{M}^- . Notice also that Ω_{relative} decreases to zero as r approaches $2M$: the fluid ends up corotating with the generators when the shell crosses the black-hole horizon.

The application considered in this subsection is a variation of an analysis presented previously by Musgrave and Lake [23]. In some respects our discussion is less general than theirs: our black hole is initially nonrotating and it does not carry an electric charge. In other respects our discussion is more general: here the shell's material is not restricted to move along the null generators; as we have seen, this generalization leads to interesting phenomena.

Cosmological phase transition

In this fourth (and last) application of the formalism, we consider an intriguing (but entirely artificial) cosmological scenario according to which the universe was expanding initially in two directions only, but was then made to expand isotropically by a sudden explosive event. This application is a simplified version of an example previously presented by Barrabès and Hogan [19].

The \mathcal{M}^- region of spacetime is the one in which the universe is expanding in the x and y directions only. Its metric is

$$ds_-^2 = -dt^2 + a^2(t)(dx^2 + dy^2) + dz_-^2, \quad (5.30)$$

and the scale factor is assumed to be given by $a(t) \propto t^{1/2}$. The cosmological fluid moves with a four-velocity $u^\alpha = \partial x^\alpha / \partial t$, and it has a density and (isotropic) pressure given by $\rho_- = p_- = 1/(32\pi t^2)$, respectively.

In the \mathcal{M}^+ region of spacetime, the universe expands uniformly in all three directions. Here the metric is

$$ds_+^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz_+^2), \quad (5.31)$$

with the same scale factor $a(t)$ as in \mathcal{M}^- , and the cosmological fluid has a density and pressure given by $\rho_+ = 3p_+ = 3/(32\pi t^2)$, respectively; this corresponds to a radiation-dominated universe.

The history of the explosive event that changes the metric from $g_{\alpha\beta}^-$ to $g_{\alpha\beta}^+$ coincides with a null hypersurface Σ . This surface moves in the positive z_\pm direction, and as we shall see, it supports a singular stress-energy tensor. The “agent” that alters the course of the universe’s expansion is therefore a null shell.

As seen from \mathcal{M}^- , the hypersurface is described by $t = z_- + \text{constant}$, and the vector $k^\alpha \partial_\alpha = \partial_t + \partial_z$ is tangent to the null generators, which are parameterized by t . Because $k^\alpha_{;\beta} k^\beta = 0$, we have that t is an affine parameter on this side of the hypersurface. The coordinates x and y are constant on the generators, and we use them, together with t , as intrinsic coordinates on Σ . We therefore have $y^a = (t, \theta^A)$, $\theta^A = (x, y)$, and the shell’s induced metric is

$$\sigma_{AB} d\theta^A d\theta^B = a^2(t)(dx^2 + dy^2). \quad (5.32)$$

The remaining basis vectors are $e_x^\alpha \partial_\alpha = \partial_x$, $e_y^\alpha \partial_\alpha = \partial_y$, and $N_\alpha dx^\alpha = -\frac{1}{2}(dt + dz_-)$. The nonvanishing components of the transverse curvature are

$$C_{AB}^- = \frac{1}{4t} \sigma_{AB}. \quad (5.33)$$

We note that on the \mathcal{M}^- side of Σ , the null generators have an expansion given by $\theta = k^\alpha_{;\alpha} = 1/t$, and that $T_{\alpha\beta} k^\alpha k^\beta = \rho_- + p_- = 1/(16\pi t^2)$, where $T^{\alpha\beta}$ is the stress-energy tensor of the cosmological fluid.

As seen from \mathcal{M}^+ , the description of the hypersurface is obtained by integrating $dt = a(t) dz_+$, and $k^\alpha \partial_\alpha = \partial_t + a^{-1} \partial_z$ is tangent to the null generators. We note that t is *not* an affine parameter on this side of the hypersurface: we have that $k^\alpha_{;\beta} k^\beta = (2t)^{-1} k^\alpha$. The remaining basis vectors are $e_x^\alpha \partial_\alpha = \partial_x$, $e_y^\alpha \partial_\alpha = \partial_y$, $N_\alpha dx^\alpha = -\frac{1}{2}(dt + a dz_+)$, and the nonvanishing components of the transverse curvature are now

$$C_{tt}^+ = \frac{1}{2t}, \quad C_{AB}^+ = \frac{1}{4t} \sigma_{AB}. \quad (5.34)$$

On this side of Σ , the generators have an expansion also given by $\theta = 1/t$ (since continuity of θ is implied by continuity of the induced metric), and $T_{\alpha\beta} k^\alpha k^\beta = \rho_+ + p_+ = 1/(8\pi t^2)$.

The fact that t is an affine parameter on one side of the hypersurface only tells us that the shell must possess a surface pressure. In fact, continuity of C_{AB} across the hypersurface implies that p is the *only* nonvanishing surface quantity. It is given by

$$p = -\frac{1}{16\pi t}, \quad (5.35)$$

the negative sign indicating that this surface quantity would be better described as a *tension*, not a pressure. The shell’s surface stress-energy tensor is $S^{\alpha\beta} = p \sigma^{AB} e_A^\alpha e_B^\alpha$. If we select observers comoving with the cosmological fluid as our preferred set of observers, then $-k_\alpha u^\alpha = 1$ and the full stress-energy tensor of the surface layer is given by $T_\Sigma^{\alpha\beta} = S^{\alpha\beta} \delta(t - t_\Sigma)$, with t_Σ denoting the time at which a given observer crosses the shell. We see that for these observers, $-p$ is the physically-measured surface tension.

Finally, we note that the expressions $-p = 1/(16\pi t)$, $\theta = 1/t$, and $[T_{\alpha\beta} k^\alpha k^\beta] = 1/(16\pi t^2)$ are compatible with the general relation $-p\theta = [T_{\alpha\beta} k^\alpha k^\beta]$ presented in Eq. (4.5). This shows that the energy released by the shell as it expands is absorbed by the cosmological fluid, whose density increases by a factor of $\rho_+/\rho_- = 3$; this energy is provided by the shell’s surface tension.

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